Exact quantum recoil in magnetic fields

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Abstract. The radiation reaction of electrons and positrons in a magnetic field is discussed using quantum electrodynamics based on the exact solution of Dirac's equation. One quantum number arising from this solution is found to depend on the position of the centre of gyration. The change in this quantum number under the emission or absorption of radiation is interpreted as a change in the centre of gyration, which leads to a classical current. A conservation law for momentum components in the xy plane may be derived from this interpretation. 'Quantum broadening' is reinterpreted as a change in the gyrocentre instead of a spread in the wavefunction. The concept of recoil is generalised to include processes involving pair production and annihilation.

1. Introduction

This paper examines the manner in which charged fermions in a magnetic field along the z axis recoil in the xy plane while emitting or absorbing radiation. The effect of the magnetic field is included exactly by using solutions of the Dirac equation

$$(i\partial_\mu \gamma^\mu + eA_\mu(x)\gamma^\mu - m)\Psi(x) = 0 \tag{1}$$

where $A_\mu(x)$ represents the 4-potential of the external field. All calculations are performed in the radiation gauge $A^0(x) = 0$, but the results are valid for all gauge choices. Natural units $c = 1$, $\hbar = 1$ are used unless otherwise stated and the charge on the electron is taken to be $-e$.

Previous classical, semiclassical and quantum results concerning recoil in a magnetic field are reviewed below. In § 2, a brief analysis of the solutions for the Dirac equation in a magnetic field is performed, based mainly on the results of Melrose and Parle (1983a, b). Section 3 contains a short discussion of gauge transformations, followed by the introduction of a coordinate transform operator, which is a generalisation of the familiar concept of a translation. This operator is used in § 4 to interpret the previously derived results for the exact recoil of an electron or positron in a magnetic field, and this interpretation is extended to include the processes of pair creation and annihilation. The emission and absorption of radiation is found to be accompanied by a change in the electric dipole moment of the system. In the final section, some applications of this new effect are considered.

It has been known for many years that when a charged particle emits or absorbs radiation, it experiences a reaction or recoil force, as required by conservation of energy-momentum. The treatment of the radiation reaction force on a classical electron

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is based on a formula of Dirac (1938). This equation has some non-physical solutions which are exponentially divergent or 'runaway', and these must be eliminated by appealing to conservation of energy. The effect of radiation reaction on the trajectory of a classical electron in a static homogeneous magnetic field has been considered by several authors. Eliezer (1947) used the Dirac prescription for the force on the electron and removed the non-physical solutions to derive the equations of motion. The result, in the non-relativistic limit, is an equiangular spiral in a plane perpendicular to the magnetic lines of force, with the electron eventually coming to rest at a point correspond-ing to what would be the centre of the orbit if radiation reaction was ignored. Eliezer found no motion of the guiding centre (GC) of the electron motion, even when most of the electron's energy is radiated away within a fraction of a complete orbit.

Plass (1961) performed a similar calculation, obtained the same result and went on to find the rate of energy loss by the electron. This rate had previously been obtained by Schwinger (1949), using a different method without solving for the electron trajec-tory, and is quoted in most modern accounts of radiation damping in a magnetic field (e.g. Suvurov and Chugunov 1973, Chugunov *et al* 1975, Landau and Lifshitz 1975). In none of these classical treatments is any evidence found of electron recoil perpen-dicular to the magnetic field.

Field-free quantum electrodynamics includes radiation reaction through the explicit conservation of 4-momentum. Only the energy and momentum parallel to the field are necessarily conserved when the exact Hamiltonian in a magnetic field is used. Sokolov and Ternov (1953, 1955, 1968) have considered the effect of the emission of synchrotron radiation on the guiding centre of an electron. They conclude that the position of the GC undergoes 'quantum broadening', i.e. the mean distance from the origin to the GC increases due to the quantised nature of the radiation field. This phenomenon was predicted by Sands (1955) and by Kolomenskii and Lebedev (1956), who described it as radial oscillations or fluctuations. This effect has also been detected experimentally by Korolev *et al* (1961).

Guiding centre drift effects such as quantum broadening are purely quantum effects which do not arise in classical treatments of synchrotron radiation or related processes. This has caused some confusion in recent papers. Lieu *et al* (1983, 1984) found an inconsistency in two results derived using semiclassical and quantum theories of cyclotron emission, and so found an apparent conflict between the predictions of the laws of conservation of linear and angular momentum. The error in this work stems from the use of a classical radiation formula which does not include guiding centre drift or radiation damping in any form (White and Parle 1985).

Quantum broadening is the only previously derived consequence of electron recoil perpendicular to the magnetic field. In the remainder of this paper, more general results are derived and several new effects are predicted. These results provide a new interpretation of quantum broadening which is invariant under time reversal and thus intuitively more satisfactory.

2. Wavefunctions in a magnetic field

Solutions of the Dirac equation in a magnetic field have been published elsewhere (Johnson and Lippmann 1950, Melrose and Parle 1983a). Different solutions are obtained depending on the gauge choice used to describe the magnetic field, but the resulting wavefunctions are related by gauge transformations (see equation (28)). The

quantum numbers needed to completely describe the wavefunctions are: $\varepsilon = \pm 1$, the sign of the energy; $n = 0, 1, 2, \ldots$, the Landau quantum number; $\sigma = \pm 1$, the spin; p_z, the parallel momentum, which here is chosen to be in the same sense as the particle velocity for both positrons and electrons; and another quantum number, denoted here by g, which contains information about the position of the guiding centre (GC) of the particle in the xy plane. The positional quantum number may be represented in many ways, two of which are described below.

In the P_y representation, the positional quantum number g is equivalent to P_y, which has continuous eigenvalues. With the gauge choice $A = (0, Bx, 0)$ for the magnetic field, referred to below as the Landau gauge, the wavefunctions have the form

$$\Psi^{\varepsilon}_{r,P_\}(x, t) = N_1^{-1/2} \exp(-\mathrm{i}\varepsilon E_q t + \mathrm{i}\varepsilon p_z z + \mathrm{i}P_y y) \begin{pmatrix} C_1 \nu_{n-1}(\xi) \\ C_2 \nu_n(\xi) \\ C_3 \nu_{n-1}(\xi) \\ C_4 \nu_n(\xi) \end{pmatrix} \tag{2}$$

with

$$\xi = \sqrt{eB}(x + P_y/eB) \qquad N_1 = L_y L_z/\sqrt{eB} \tag{3a, b}$$

$$\nu_n(\xi) = (\sqrt{\pi} 2^n n!)^{-1/2} H_n(\xi)\, e^{-\xi^2/2} \tag{4}$$

where $r = \{n, \sigma, p_z\}$ and one has $C^\dagger C = 1$. The C_i are determined by the requirement that the wavefunction satisfy the Dirac equation and is an eigenvector of the particular spin operator selected. The quantum number P_y is related to the x position of the GC, as derived from the relation

$$\langle q|x|q \rangle = -P_y/eB = x_0. \tag{5}$$

Now the expectation value for x^2 for a particle in state q is

$$\langle q|x^2|q \rangle = \langle q|(x - x_0)^2|q \rangle + x_0^2$$
$$= (1/eB)(n - \tfrac{1}{2}\Sigma_{zq} + P_y^2/eB) \tag{6}$$

where Σ_{zq} is the expectation value of the spin angular momentum operator along the z axis

$$\Sigma_{zq} = \langle q|\Sigma_z|q \rangle = |C_1|^2 + |C_3|^2 - |C_2|^2 - |C_4|^2. \tag{7}$$

One has $-1 \leqslant \Sigma_{zq} \leqslant 1$ and $\Sigma_{zq} = -1$ in the ground state $n = 0$. Now consider a classical electron orbiting with radius R around a point with fixed x coordinate x_0. The mean-square x coordinate for the particle is

$$\langle x^2 \rangle = \tfrac{1}{2} R^2 + x_0^2 \tag{8}$$

and, comparing with (6), the mean orbital radius for a quantum electron may be identified:

$$R_q^2 = (2/eB)(n - \tfrac{1}{2}\Sigma_{zq}) \tag{9}$$

which in the classical limit $n \to \infty$ reproduces the usual result $R = p_\perp/eB$. A particle in an eigenstate of the $\hat{P}_y$ operator thus has a guiding centre whose x coordinate is fixed and whose y coordinate is undetermined (see figure 1(a)).

In the s representation, the positional quantum number g is equivalent to s, which has non-negative integral eigenvalues. With the gauge choice $A = \tfrac{1}{2} B(-y, x, 0)$ for the
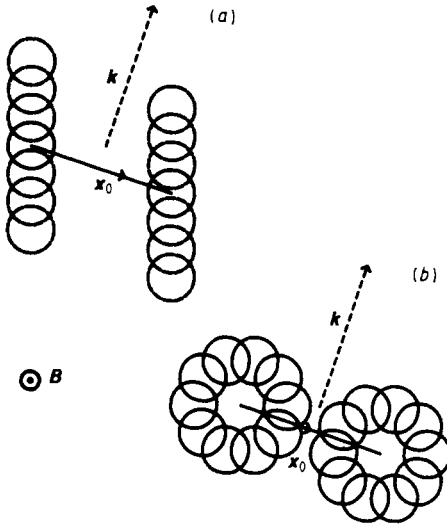
Figure 1. Recoil for an electron initially in an eigenstate of the positional quantum number, emitting photons with a total 3-momentum of k. (a) P_y representation, (b) s representation. In the latter case, the centre of symmetry has been translated from the origin to x_0^+. The radii of the circles, representing the classical orbital radius of the electron, are changed by the emission or absorption of radiation, but the locus of the GC remains constant in size while being translated, as the numerical value of g is the same on both sides of (60).

magnetic field, referred to below as the cylindrical gauge, the wavefunctions have the form

$$\Psi_{r,s}^{\varepsilon}(\boldsymbol{x}, t) = N_2^{-1/2} \exp[-i\varepsilon E_q t + i\varepsilon p_z z + i\phi(n-s-\tfrac{1}{2})] \begin{pmatrix} C_1 \, e^{-i\phi/2} J_{n-s-1}^s(\rho) \\ C_2 \, e^{+i\phi/2} J_{n-s}^s(\rho) \\ C_3 \, e^{-i\phi/2} J_{n-s-1}^s(\rho) \\ C_4 \, e^{+i\phi/2} J_{n-s}^s(\rho) \end{pmatrix} \tag{10}$$

with

$$\rho = \tfrac{1}{2} eBr^2 \qquad N_2 = 2\pi L_z / eB. \tag{11a, b}$$

The J_ν^n functions are related to the generalised Laguerre polynomials

$$J_\nu^n(x) = [n!/(n+\nu)!]^{1/2} e^{-x/2} x^{\nu/2} L_n^\nu(x) \tag{12}$$

and only exist for $n \geq 0$, $n + \nu \geq 0$. Note that the C_i are chosen to be the same in both representations.

The value of the radial quantum number s is related to the distance of the centre of gyration from the z axis. The expectation value for the square distance of the particle from the axis is given by

$$\langle q | r^2 | q \rangle = (2/eB)(n + s + \tfrac{1}{2} - \tfrac{1}{2} \Sigma_{zq})$$
$$= R_q^2 + (2/eB)(s + \tfrac{1}{2}) \tag{13}$$

where (9) has been used to derive the last line. Now consider the same quantity for a classical electron with an orbit of radius R around a point at distance a from the origin:

$$\langle r^2 \rangle = R^2 + a^2. \tag{14}$$

Thus the distance from the centre of gyration to the origin for a quantum electron is identified as

$$a_q^2 = (2s+1)/eB. \tag{15}$$

This corrects the analysis by Sokolov and Ternov (1968), who neglected the constant term. The locus of the guiding centre is thus a circle of radius a_q centred on the origin (see figure 1(b)). No information is known about the position angle of the centre on this locus. Quantum broadening is easily demonstrated using this representation. The scattering matrix element for the process where an electron passes from state q to state q' while emitting photons of total 3-momentum $\boldsymbol{k}$ has the form (Melrose and Parle 1983b)

$$S_{q'q}(\boldsymbol{k}) = (\mathrm{i}\,\mathrm{e}^{\mathrm{i}\phi})^{s'-s}J_{s'-s}^s\left(\frac{k_\perp^2}{2eB}\right)T_{n'\sigma'p_z', n\sigma p_z}(\boldsymbol{k}). \tag{16}$$

If the electron is initially in an eigenstate of $\hat{s}$ with eigenvalue s_i, an elementary calculation shows that the mean-square distance of the GC of the electron orbit from the origin increases:

$$\langle a_f^2 - a_i^2 \rangle = \frac{2}{eB}\langle s_f - s_i \rangle = \left(\frac{k_\perp}{eB}\right)^2 \tag{17}$$

and so quantum broadening appears to takes place.

The eigenfunctions of $\hat{s}$ are particularly important because they are also eigenfunctions of the total angular momentum operator $\hat{J}_z$, with eigenvalues

$$J_z = (n - s - \tfrac{1}{2}). \tag{18}$$

This includes both orbital and spin contributions. These particular wavefunctions have this property because of their cylindrical symmetry around a line parallel to the magnetic field passing through the origin. More generalised versions of these wavefunctions are considered below where the 'centre of symmetry' can lie anywhere on the xy plane.

A wavefunction in one gauge can be transformed into the other gauge by performing the appropriate transform (see (28)). In addition, solutions in different representations must also be related, as the set of quantum numbers in each representation is complete. The wavefunctions (2) and (10) are related by

$$\Psi_{r, P_y}^\varepsilon(\boldsymbol{x}) = \sum_{s=0}^\infty \exp(-\tfrac{1}{2}\mathrm{i}eBxy)a(P_y, s)\Psi_{r,s}^\varepsilon(\boldsymbol{x}) \tag{19}$$

$$\Psi_{r,s}^\varepsilon(\boldsymbol{x}) = L_y \int \frac{\mathrm{d}P_y}{2\pi} \exp(\tfrac{1}{2}\mathrm{i}eBxy)a(P_y, s)\Psi_{r, P_y}^\varepsilon(\boldsymbol{x}) \tag{20}$$

with $r \equiv \{n, p_z, \sigma\}$ and where the expansion coefficients are given by

$$a(P_y, s) = \left(\frac{2\pi}{L_y\sqrt{eB}}\right)^{1/2} \nu_s(P_y/\sqrt{eB}). \tag{21}$$

Quantities which depend on the form of the positional quantum number g are termed 'representation dependent' and need not be gauge dependent.

The result used here is the form of the factor associated with each fermion line which depends on the positional quantum number and which is obtained by summing over the values of g for each virtual state on the line (Melrose and Parle 1983b, equation 24). For an open fermion line with incoming state (initial electron or final positron)

quantum numbers $\{\varepsilon, r, g\}$ and outgoing state (final electron or initial positron) quantum numbers $\{\varepsilon', r', g'\}$, and with total photon momenta associated with the vertices on the line of k^μ, the factor is

$$G_{P'_i P_i}(k) = \frac{2\pi}{L_y} \delta(P_y - P'_y - k_y) \exp\left(\frac{ik_x}{2eB}(P_y + P'_y)\right) \tag{22}$$

$$G_{s's}(k) = (i\, e^{i\phi})^{s'-s} J^s_{s'-s}(k_\perp^2/2eB) \tag{23}$$

in the P_y and s representations respectively. A closed fermion loop which has no external fermion quantum numbers associated with it is necessarily independent of the positional quantum number and the representation, and summing over the intermediate values of the positional quantum number yields a factor of

$$\frac{2\pi}{L_x} \delta(k_x) \frac{2\pi}{L_y} \delta(k_y). \tag{24}$$

These results were derived *in vacuo*, but may be readily generalised to media which have fermion occupation numbers which are independent of the positional quantum number. Homogeneous media, which are the same wherever they are observed, fall into this category.

3. Gauge and coordinate transforms

Gauge invariance requires that any physical quantity must be left unchanged by a gauge transformation, which in this context is a unitary contact transformation of the wavefunctions and operators. Solutions of the Dirac equation in two different gauges are related by such a transformation. If the external 4-potentials in the two gauges are related by

$$A_2^\mu(x) = A_1^\mu(x) + \partial^\mu f(x) \tag{25}$$

then the kinetic momenta are related by

$$\Pi_2^\mu = (i\partial^\mu + eA_2^\mu(x)) = S_{21}(x)\Pi_1^\mu S_{21}^{-1}(x) \tag{26}$$

where $S_{21}(x)$, the contact transformation, is given by

$$S_{21}(x) = S_{12}^{-1}(x) = \exp(ief(x)). \tag{27}$$

Note that A_1 and A_2 only determine f to within a constant. Solutions of the Dirac equation in the two gauges are related by

$$\Psi_2(x) = S_{21}(x)\Psi_1(x). \tag{28}$$

As f is determined to within a constant, S_{21} can include an undetermined phase factor. When all wavefunctions are transformed according to (28) and all operators according to (26), then all amplitudes and hence all physical results remain invariant.

Any bilinear form constructed from the fermion wavefunctions evaluated at the same point in spacetime is invariant under gauge transformations because of the unitary property $S(x)S^\dagger(x) = 1$ which transforms of the type (27) satisfy. One such bilinear form is the factor $\bar{\Psi}_{q'}(x)\gamma^\mu\Psi_q(x)$ appearing at each vertex of the Feynman diagram, and hence the diagram amplitude is invariant under the restricted class of gauge transformations considered here. In particular, $G_{g'g}(k)$ given by (22) and (23) is dependent only on the representation of g and not on the choice of gauge.

In the following calculations, gauge transformations of the potential representing the static magnetic field are used in the construction of coordinate transformations. The fermion wavefunctions in a magnetic field are generally not covariant under the translation

$$x'^{\mu} = x^{\mu} - x_0^{\mu} \tag{29}$$

and hence the generator of such a translation does not represent a conserved quantity. The reason for the lack of covariance lies in the fact that the 4-potential representing the magnetic field is a function of the coordinates relative to some fixed origin O and a translation of the form (29), equivalent to choosing a new origin O' at x_0, also causes a change of gauge.

The translation (29) may be represented by an operator $T(x_0)$ with the properties

$$T(x_0)f(x) = f(x - x_0) \tag{30a}$$

$$T(x_0)T(x_0') = T(x_0 + x_0') \tag{30b}$$

which operates on all wavefunctions to the right. A coordinate transformation operator $R(x, x_0)$ is to be constructed which causes a translation without creating a gauge transformation. One writes

$$R(x, x_0) = S(x, x_0)T(x_0) = T(x_0)S(x + x_0, x_0) \tag{31}$$

where $S(x, x_0)$ is a gauge transformation chosen to cancel the transformation caused by $T(x_0)$. There is a requirement on S that the kinetic momenta Π^{μ} (and hence the Hamiltonian, parallel momentum and spin operators) are left invariant under the coordinate transformation

$$R(x, x_0)\Pi^{\mu}R^{-1}(x, x_0) = \Pi^{\mu}. \tag{32}$$

Any sensible coordinate transform must satisfy the identity and inverse relations

$$R(x, 0) = 1 \qquad R^{-1}(x, x_0) = R(x, -x_0) \tag{33a, b}$$

which may be expressed as conditions on the gauge transformation S. Writing S in the form

$$S(x, x_0) = \exp(ief(x, x_0)) \tag{34}$$

the identity condition becomes

$$f(x, 0) = 0 \tag{35}$$

and the inverse condition may be written as

$$f(x - x_0, -x_0) = -f(x, x_0). \tag{36}$$

The coordinate transform R operating on a wavefunction to the right yields

$$R(x, x_0)\Psi(x) = \Psi(x - x_0)S(x, x_0). \tag{37}$$

Taking the Dirac conjugate of both sides and using $(33b)$, one obtains the corresponding relation

$$\overline{R(x, x_0)\Psi(x)} = \bar{\Psi}(x - x_0)S^{\dagger}(x, x_0) = \bar{\Psi}(x)R^{-1}(x, x_0) \tag{38}$$

where the transform operator operates to the left.

Now consider the invariance condition (32). The left-hand side is

$$S(x, x_0)T(x_0)(i\partial^{\mu} + eA^{\mu}(x))S(x, -x_0)T(-x_0) = i\partial^{\mu} + eA^{\mu}(x - x_0) + e\partial^{\mu}f(x, x_0) \tag{39}$$

and so $f(x, x_0)$ must satisfy the requirement

$$\partial^\mu f(x, x_0) = A^\mu(x) - A^\mu(x - x_0). \tag{40}$$

Equation (40) may be integrated to find f to within a function of the parameter x_0. This function may be determined by applying the condition (36). The integral is path independent as the right-hand side of (40) is irrotational. The result may be written in the form

$$f(x, x_0) = \int_{x_0/2}^{x} dx'_\mu (A^\mu(x') - A^\mu(x' - x_0)). \tag{41}$$

Note that, using (41), it can be shown that R is itself gauge covariant and satisfies a relation equivalent to (26).

The above arguments are valid for potentials describing all kinds of external fields. From now on, only a static homogeneous magnetic field is considered. As the momentum components in the temporal and z direction are conserved, it is convenient to restrict the translations to lie in the xy plane, i.e. $x_0^\mu = (0, x_0, y_0, 0)$. A property of the coordinate transform operator R in a magnetic field is that two operators with non-parallel x_0 do not commute. The operators satisfy the gauge-independent relation

$$R^{-1}(x, x_2) R^{-1}(x, x_1) R(x, x_2) R(x, x_1) = \exp(ieB(x_1 \times x_2)_\parallel). \tag{42}$$

By construction, the operators R may be used as generators of infinitesimal translations which leave the Hamiltonian invariant and hence correspond to a conserved momentum. This momentum, called pseudomomentum by Avron *et al* (1978) and generalised momentum by Herold *et al* (1981), has a component in the direction of the unit vector a given by

$$\begin{aligned}
a \cdot P &= -i \left[\frac{\partial}{\partial \alpha} R(x, \alpha a) \right]_{\alpha=0} \\
&= -i a \cdot \nabla + e \left[\frac{\partial}{\partial \alpha} f(x, \alpha a) \right]_{\alpha=0} \\
&= -i a \cdot \nabla - e \int^{x} a_j \frac{\partial}{\partial x'_j} A_k(x') \, dx'_k
\end{aligned} \tag{43}$$

and hence

$$P_j = -i \frac{\partial}{\partial x_j} - e \int^{x} \frac{\partial}{\partial x'_j} A_k(x') \, dx'_k. \tag{44}$$

As the operators R do not commute for translations in different directions, the commutator for the momentum components in the x and y directions is non-zero:

$$[P_x, P_y] = -ieB. \tag{45}$$

These operators are mentioned here only for completeness and are not used below. The theory of quantum recoil depends on the existence of the operator R which generates *finite* translations.

This section concludes with the calculation of the matrix elements of the coordinate transform operator R:

$$\begin{aligned}
R_{q'q}(x_0) &= \int dx \, \Psi_{q'}^\dagger(x) R(x, x_0) \Psi_q(x) \\
&= \int dx \, S(x, x_0) \Psi_{q'}^\dagger(x) \Psi_q(x - x_0).
\end{aligned} \tag{46}$$

As R has been defined to leave the Hamiltonian, parallel momentum and spin operators invariant, the matrix element is zero unless the quantum numbers ε, n, p_z and σ are the same for both wavefunctions in (46). One writes

$$R_{q'q}(x_0) = \delta_{\varepsilon'\varepsilon}\delta_{r'r}R_{g'g}(x_0) \tag{47}$$

which leaves only the reduced matrix elements to be calculated. Once these are known, the completeness relation for the solutions of the Dirac equation may be used to expand the translated wavefunction in terms of the original wavefunctions:

$$R(x, x_0)\Psi_{r,g}^{\varepsilon}(x) = \sum_{g'} R_{g'g}(x_0)\Psi_{r,g'}^{\varepsilon}(x). \tag{48}$$

This is performed below for the wavefunctions (2) and (10).

In the Landau gauge, the transformation operator is given by

$$R(x, x_0) = \exp[-ieBx_0(y - \tfrac{1}{2}y_0)]T(x_0) \tag{49}$$

calculated using (41), and integrating (46) yields the result

$$R_{P'_y, P_y}(x_0) = (2\pi/L_y)\delta(P'_y - P_y + eBx_0)\exp[-iy_0(P_y - \tfrac{1}{2}eBx_0)]. \tag{50}$$

The transformed wavefunction is then given by

$$R(x, x_0)\Psi_{r, P_y}^{\varepsilon}(x) = \exp[-iy_0(P_y - \tfrac{1}{2}eBx_0)]\Psi_{r, P_y - eBx_0}^{\varepsilon}(x). \tag{51}$$

In the cylindrical gauge, the transform operator is given by

$$R(x, x_0) = \exp[\tfrac{1}{2}ieB(xy_0 - yx_0)]T(x_0). \tag{52}$$

In this case, rather than integrate (46) directly, it is more convenient to use the representation (20) for the wavefunctions in the s representation. After some algebra, one obtains

$$R_{s's}(x_0) = L_y \int \frac{\mathrm{d}P_y}{2\pi} L_y \int \frac{\mathrm{d}P'_y}{2\pi} \frac{2\pi}{L_y\sqrt{eB}} R_{P'_y P_y}(x_0) \nu_{s'}(P'_y/eB)\nu_s(P_y/eB). \tag{53}$$

Performing the integrals in (53), one obtains the final result

$$R_{s's}(x_0) = (-e^{i\phi_0})^{s'-s}J_{s'-s}^{s}(\rho_0) \tag{54}$$

$$x_0 + iy_0 = \left(\frac{2\rho_0}{eB}\right)^{1/2}e^{i\phi_0}. \tag{55}$$

The transformed wavefunction is then given by

$$R(x, x_0)\Psi_{r,s}^{\varepsilon}(x) = \sum_{s'} (-e^{i\phi_0})^{s'-s}J_{s'-s}^{s}(\rho_0)\Psi_{r,s'}^{\varepsilon}(x). \tag{56}$$

The concept of a translation has now been generalised to allow the construction of a coordinate transformation operator which leaves the Hamiltonian and other operators invariant. An explicit representation of this operator has been found and its matrix elements calculated. Note that the final form of the matrix elements depends on the representation but not on the gauge.

The wavefunctions $R(x, x_0)\Psi(x)$ and $\bar{\Psi}(x)R^{-1}(x, x_0)$ represent wavefunctions which have been translated in space but still satisfy the Dirac equation in the original gauge. Using the reduced matrix elements $R_{g'g}(x_0)$, the translated wavefunctions may be expressed as linear combinations of the original wavefunctions. The positional quantum numbers g of the translated wavefunctions must be interpreted relative to the new origin determined by the translation. In particular, the wavefunction $R(x, x_0)\Psi_{rs}^{\varepsilon}(x)$ has cylindrical symmetry around the axis passing through x_0^{μ}.

4. Quantum recoil

In Melrose and Parle (1983b), a procedure was developed for the calculation of Feynman amplitudes in the momentum representation. In particular, the dependence of the amplitude on the values of the positional quantum number is given by a factor (22) or (23) for each open fermion line, where g and g' are the positional quantum numbers associated with the incoming fermion edge and the outgoing fermion edge for the line. It is this function which determines how fermions recoil in the xy plane.

Consider first the case of an open fermion line whose incoming edge represents an electron in the initial state with quantum numbers q_1 and whose outgoing edge represents an electron in the final state with quantum numbers q_2. Let the total 4-momentum associated with the vertices on the line be given by k^μ. If $S_{q_1 \to q_2}(k^\mu)$ is the amplitude for the electron to scatter from q_1 to q_2, then the final-state wavefunction is given by

$$\Psi_f^+(x) = \sum_{q_2} \Psi_{q_2}^+(x) S_{q_1 \to q_2}(k^\mu)$$

$$= \sum_{r_2} \Phi_{r_2, g_1}^+(x; k) S_{r_1 \to r_2}(k^\mu) \tag{57}$$

where $S_{r_1 \to r_2}(k^\mu)$ is a reduced scattering matrix element which is independent of the positional quantum numbers g_1, g_2 and

$$\Phi_{r_2, g_1}^\varepsilon(x; k) = \sum_{g_2} \Psi_{r_2, g_2}^\varepsilon(x) G_{g_2 g_1}(k). \tag{58}$$

Let a displacement vector in the xy plane for an electron ($\varepsilon = +1$) be defined by

$$[x_0^+]^\mu = \frac{1}{eB}(0, k \times b) = \frac{1}{eB}(0, k_y, -k_x, 0) \tag{59}$$

where b is a unit vector pointing in the direction of the magnetic field. Then by comparison of the expressions (22) and (23) for the representation-dependent part of the amplitude, and (51) and (56) for the coordinate transform operator, one obtains the (representation-independent) result

$$\Phi_{r_2, g_1}^+(x; k) = R(x, x_0^+) \Psi_{r_2, g_1}^+(x). \tag{60}$$

This may be interpreted as saying that the final-state wavefunction for fixed r_2 is given by $\Psi_{r_2, g_1}^+(x)$ translated by x_0^+. Note that the numerical value of the positional quantum number is the same on the right-hand side of (60) as it was before the electron was scattered, and so the locus of the guiding centre has the same form before and after scattering, but has been translated by that amount. Hence the electron GC has recoiled in the xy plane, and the final-state wavefunction may be written in the form

$$\Psi_f^+(x) = R(x, x_0^+)\left(\sum_{r_2} \Psi_{r_2, g_1}^+(x) S_{r_1 \to r_2}(k^\mu)\right). \tag{61}$$

This recoil is shown in figure 1 for the case of the electron initially in an eigenstate of $\hat{P}_y$ and $\hat{s}$.

For the case where the final quantum numbers r_2 are completely determined, the final-state wavefunction is given, to within a phase factor, by

$$\Psi_f^+(x) = R(x, x_0^+)\Psi_{r_2 g_1}^+(x) \tag{62}$$

where the reduced matrix element no longer appears because the wavefunction Ψ_f^+ is normalised to unity and now indicates through the Fermi golden rule the rate of transition from the state described by $\Psi_{r_i g_i}^+$ to Ψ_f^+.

As any wavefunction can be represented an a superposition of eigenfunctions of $\hat{P}_y$ or $\hat{s}$, equation (61) is representation independent and so the interpretation of $G_{g'g}(k)$ as representing the recoil of the GC in the xy plane is valid in all gauges and representations of the positional quantum number.

This interpretation of quantum recoil gives a physical justification for the apparent outward drift of the guiding centre referred to above as quantum broadening. As in that analysis, assume that the electron is initially in a state r_i, s_i and radiates photons of 4-momentum $k_1^\mu, \ldots, k_n^\mu$, ending in a state with quantum numbers r_f. Writing the recoil vector due to the emission of the ith photon as

$$x_i = (1/eB)\mathbf{k}_i \times \mathbf{b} \tag{63}$$

then the final state of the electron is given, to within a phase factor, by

$$\Psi_f^+(x) = R(x, x_n) \ldots R(x, x_1)\Psi_{r_i s_i}^+(x)$$

$$= \exp\left(\tfrac{1}{2} i e\mathbf{B} \cdot \sum_{i<j} (\mathbf{x}_i \times \mathbf{x}_j)\right) R(x, x_1 + \ldots + x_n)\Psi_{r_i s_i}^+(x). \tag{64}$$

Before emitting the first photon, the GC was at a distance

$$a_i = \left(\frac{2s_i + 1}{eB}\right)^{1/2} \tag{65}$$

from the origin, while after emitting n photons, the GC lies at a distance a_i from the point

$$\mathbf{X}_n = \sum_{i=1}^{n} \mathbf{x}_i = \frac{1}{eB}\left(\sum_{i=1}^{n} \mathbf{k}_i\right) \times \mathbf{b}. \tag{66}$$

The wavefunction itself does not spread out, rather the point $\mathbf{X}_n$ about which the wavefunction has cylindrical symmetry moves away from the origin. The mean distance of the GC from the origin after the emission of n photons is given by

$$R_n^2 = a_i^2 + X_n^2 = a_i^2 + \left(\sum_{i=1}^{n} \frac{k_{\perp i}}{eB}\right)^2 \tag{67}$$

which reproduces the result (17). If the emission of successive photons is essentially independent, then the trajectory of the centre of symmetry X_n through the xy plane as n increases is described by a random walk in two dimensions, and hence the expectation value of $|X_n|$ increases with n, while the expectation value of (X_n) remains at the origin.

The result (62) for the recoil of an electron may be readily generalised to the case of an open fermion line whose incoming and outgoing edges represent a positron in the final and initial states respectively. The conjugate wavefunction is the appropriate one to represent a positron because it has the correct time dependence for a real particle and is associated, in the operator $\bar{\psi}$, with the positron annihilation operator. One obtains the corresponding relation to (60):

$$\bar{\Phi}_{r_2 g_1}^-(x; k) = \overline{R(x, x_0^-)\Psi_{r_2 g_1}^-(x)} = \bar{\Psi}_{r_2 g_1}^-(x)R^{-1}(x, x_0^-) \tag{68}$$

where the positron ($\varepsilon = -1$) displacement vector is

$$[x_0^-]^\mu = -(1/eB)(0, \mathbf{k} \times \mathbf{b}) = -[x_0^-]^\mu. \tag{69}$$

As in the case of an electron, this result indicates that the GC of the positron recoils in the xy plane by the amount x_0^-. The displacement is equal in magnitude but opposite in direction to that of the electron, as is required by the *CPT* invariance of quantum electrodynamics.

The concept of quantum recoil can be generalised to include fermion lines representing pair annihilation and creation. Consider first the case of an open line whose incoming edge represents a positron with quantum numbers q and whose outgoing edge an electron with quantum numbers q', both in the final state. If the total photon 4-momentum associated with the vertices on the line is k^μ, then the wavefunction of the created pair, written as an outer product of single-particle wavefunctions, has the form

$$\Psi_f^+(x') \times \bar{\Psi}_f^-(x) = \sum_{r'r} S_{r'r}(k^\mu) \sum_{g'g} G_{g'g}(k) \Psi_{r'g'}^+(x') \times \bar{\Psi}_{rg}^-(x)$$

$$= \sum_{r'r} S_{r'r}(k) \sum_g \left(R(x', x_0^+) \Psi_{r'g}^+(x') \times \bar{\Psi}_{rg}^-(x) \right) \tag{70}$$

where the displacement vector is given by (59). Equation (70) involves a sum over the quantum number g but is otherwise independent of the positional quantum numbers of the final particles. This means that there is no information about the position in the xy plane of any one particle, and the GC is evenly distributed. There is, however, a known relation between the positions of the guiding centres of the two particles. The loci of the two GC are identical, except that that of the electron is displaced from that of the positron by x_0^+. Hence a measurement of the locus of the GC of the positron would determine that of the electron to be at a fixed distance x_0^+ away.

The result (70) is not unique: the final summand may be written as

$$\Psi_{r'g}^+(x') \times \bar{\Psi}_{rg}^-(x) R^{-1}(x, x_0^-) \tag{71}$$

or in many equivalent forms, all of which have the physical interpretation that the loci of the GC of the two particles are identical in form but displaced from each other by a fixed vector

$$\langle x_{\text{electron}} - x_{\text{positron}} \rangle_\perp = (1/eB) k \times b \tag{72}$$

while the corresponding results for electron and positron scattering derived above have the form

$$\langle x_{\text{final}} - x_{\text{initial}} \rangle_\perp = (\varepsilon/eB) k \times b. \tag{73}$$

The relations (72) and (73) have been derived under the assumption that there is no post selection of the value of the positional quantum number and hence the final wavefunction is a superposition of different eigenfunctions with relative amplitudes determined only by the process itself. Such post selection occurs when an experiment is performed to determine g, which disturbs the system unless it happens to be in an eigenstate of $\hat{g}$. Measurement of the values of the quantum numbers r need not, in principle, disturb the dependence of the system on g.

As QED is invariant under time reversal, a result equivalent to (72) can be derived for the process of pair annihilation when there is no preselection of the positional quantum number (Aharonov *et al* 1964). When the initial wavefunctions include all values of the positional quantum number with equal amplitude, the relation between the positions of the fermion guiding centres and the total photon momentum is given by

$$\langle x_{\text{electron}} - x_{\text{positron}} \rangle_\perp = -(1/eB) k \times b. \tag{74}$$

In particular, this relation holds when the transition rate averaged over the positional quantum numbers is used (Melrose and Parle 1983b).

At this point it is clear that the representation-dependent part $G_{g'g}(k)$ of the amplitude for an open fermion line represents the generalised recoil of the fermion in a direction perpendicular to the magnetic field. Although the guiding centre of a particle's orbit can never be precisely determined, the change in the locus of the guiding centre is given by the appropriate relation (72)-(74).

For a process represented by Feynman diagrams with more than one open fermion line, one can write down a relationship connecting the positions of all fermions in the initial and final states with the total momenta of the external photons. Using primes to denote particles in the final state, one finds

$$\left\langle \left(\sum_i x'_i - \sum_i x_i \right)_{\text{electron}} - \left(\sum_i x'_i - \sum_i x_i \right)_{\text{positron}} \right\rangle_\perp = (1/eB) \left(\sum_i k'_i - \sum_i k_i \right) \times b. \tag{75}$$

The electric dipole moment for a system of classical point particles with coordinates x_i and charges q_i is given by

$$P \equiv \sum_i q_i x_i. \tag{76}$$

Except in the case of a system with no net charge $\sum_i q_i = 0$, the dipole moment is defined only to within a constant vector which depends on the choice of the coordinate origin. If at a later time the particle coordinates are given by x'_i and charges by q'_i, where the total charge of the system may not vary, the change in the system electric dipole moment is

$$\Delta P = \sum_j q'_j x'_j - \sum_i q_i x_i \tag{77}$$

which is well defined and independent of the origin. An analogue of the electric dipole moment may be constructed for a quantum system by replacing the point particle coordinates by the quantum expectation value of the particle position. Considering a unit volume and multiplying both sides of equation (75) by the electronic charge $-e$ and using (77), one obtains

$$\Delta P_\perp = -(1/B)\Delta K \times b \tag{78}$$

where P is now the polarisation density and

$$\Delta K = \sum_i k'_i - \sum_i k_i \tag{79}$$

is the increase in the total 3-momentum density of the radiation field. The interpretation of this result is discussed below.

5. Applications of quantum recoil

In this section, the physical effects of quantum recoil are considered. One such effect, the phenomenon of quantum broadening, has already been discussed in the previous section. Other consequences discussed below include a recoil correction to the angular momentum of a cyclotron photon and the existence of radiation induced currents and charge separation in plasmas. In this section, $\hbar$ and c are included in the equations.

5.1. Angular momentum of cyclotron photons

The standard result for the angular momentum of a cyclotron photon emitted at the λth harmonic is $J_{z,\lambda} = \lambda \hbar$. The angular momentum is measured with respect to the origin, which is assumed to be the guiding centre of the electron orbit. Now consider an electron in a state r, s emitting a cyclotron photon and ending in a state r', s', with s' undetermined. The expectation value of s' is given by (17) and, using the result (18) for the angular momentum of the electron, the expectation value of the angular momentum of the cyclotron photon is

$$\langle J_{z,\gamma} \rangle = \hbar \langle n - s - n' + s' \rangle = \lambda \hbar + (\hbar k_\perp)^2 / 2eB \tag{80}$$

where the second term is a recoil correction to the standard value for the angular momentum (White and Parle 1985). In the non-relativistic limit with $k_\perp \approx \lambda \Omega_c / c$, this correction is of order $E_\gamma / 2E_q$.

5.2. Radiation induced currents and charge separation

According to equation (78), the emission and absorption of radiation leads to a change in the electric dipole moment density and hence a charge separation which depends on the change in the total perpendicular momentum density of the radiation field. Taking the vector product of both sides of this equation with B, and including all factors of $\hbar$ and c, one obtains

$$\Delta S_\perp = c^2 \Delta P \times B. \tag{81}$$

This has the form of an equation in classical electrodynamics, as it is independent of $\hbar$, and S is just the Poynting vector.

Note that the parameters representing the mass and charge of the fermion have been eliminated from equation (81), implying that it is valid for all charged point-like particles with spin $\frac{1}{2}$. It is also independent of any radiative corrections in quantum electrodynamics when the bare parameters are replaced by the observed ones. A possible topic of future research is to determine whether this relation holds in general.

Taking the derivative of (81) with respect to time yields the result

$$\frac{1}{c^2} \frac{d}{dt} S_\perp = J_{rad} \times B \tag{82}$$

where J_{rad} denotes a current which is induced in the plasma by radiation reaction. Interpreting (82) as a relation in classical plasma dynamics, the left-hand side is the force exerted on the radiation field by the rest of the system. The right-hand side is the force that the ambient magnetic field would exert on a current density J_{rad}. The equality existing between these two quantities may be taken to mean that the force required to increase the momentum density S/c^2 of the radiation field is supplied by the $J \times B$ force on the radiation induced current: equivalently, the reaction force $-\dot{S}/c^2$ of the radiation acting on the rest of the system is balanced by the $J_{rad} \times B$ force, and so the net force acting on the current is zero. Equation (82) then has the significance of a conservation law: the momentum of the radiation field is supplied by the magnetic field acting on J_{rad}.

If regarded naively, equation (82) seems to imply that the current becomes infinite at low fields. This is not expected for two reasons. The first reason is that, for weak fields, scattering processes in the magnetic field which involve significant change in

the perpendicular momentum of the system are of order greater than one in the magnetic field strength, and so the left-hand side of (82) will tend to zero faster than $\boldsymbol{B}$. The second reason is that the implicit assumption that the magnetic field is rigid (and hence not affected by the processes going on within it) will break down if the current increases, as the induced current will in turn induce an additional magnetic field.

No assumptions about the properties of the medium have been made in deriving the equations of the last two sections other than that it is homogeneous. In particular, no assumptions have been made about the wave properties in the derivation of the vertex function and its representation-dependent part $G_{g'g}$. The formula (82) for the current induced by radiation reaction must therefore hold for all wave modes in a homogeneous medium which is composed of spin-$\frac{1}{2}$ particles which may be treated as point-like, such as an electron–positron or electron–proton plasma. In such a plasma, (82) indicates that there is an induced current associated with the damping of waves. If the power density dissipated by waves of wavenumber k in the mode M is $P_M(k)$, then the response of the plasma to the wave includes a current density induced by a radiation reaction given by

$$J_{\text{rad}} = \sum_{M,k} \frac{P_M(\boldsymbol{k})}{\omega_M(\boldsymbol{k})B} \, \boldsymbol{k} \times \boldsymbol{b} \tag{83}$$

where $\omega_M(\boldsymbol{k})$ is the relevant solution of the dispersion relation.

The quantum recoil effect discussed above applies to magnetised plasmas composed of charged spin-$\frac{1}{2}$ particles. In the remainder of this section, the extension of this effect to electrons in crystals is considered. The lattice potential may be included by using the ansatz of replacing the free electron mass m by the cyclotron effective mass m^*, which may be determined by cyclotron resonance experiments. Such an approach has been used (Laughlin 1981) with the non-relativistic Hamiltonian to explain the quantised Hall effect. As the equations for the radiation induced current are independent of the mass of the fermions, they are still valid when the lattice potential is included.

Any complete theory of quantum recoil in crystals at non-zero temperatures must consider the electron–phonon interaction. The recoil equations can be generalised to include the change in phonon momentum on the left-hand side of (81). Alternatively, phonon–electron interactions can be shown to be ineffective under many conditions of interest. Such is the case for experiments carried out in strong laboratory fields and at very low temperatures. In such an arrangement, when the energy of the few phonons present is much less than the cyclotron energy, scattering from phonons cannot occur (Stormer and Tsui 1983).

Recent experiments provide some justification that the predictions of QED in a magnetised vacuum may be generalised to electrons in crystals, with only minor modification (such as the effective mass). In strong laboratory fields ($B \geqslant 15$ T) and helium temperatures, the electron gas is fully quantised and the behaviour of the electrons is fully determined by the magnetic field (von Klitzing *et al* 1980, Stormer *et al* 1983, Halperin 1982, Davies and Pepper 1983). Under these conditions, the quantum recoil effect should be observable. Current densities induced by radiation are small, however: for the quoted field strength and with a power dissipated per unit volume of P W m^{-3}, the maximum induced current density is only $2.2 \times 10^{-10} P$ A m^{-2}.

In this paper, the recoil of a charged particle due to the component of photon momentum perpendicular to the magnetic field has been identified and interpreted. This leads to a more accurate analysis of such phenomena as the quantum broadening of the cyclotron orbit and the angular momentum of the cyclotron photon. Quantum

recoil is readily generalised to the classical regime and several new effects such as the current induced by radiation damping in magnetised plasmas and crystals have been presented.

Acknowledgments

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